Lecture 12

Disjoint-Set Data Structure (contd.)

Source: Introduction to Algorithms, CLRS

Union on Disjoint-Sets as Trees using Rank

Union on Disjoint-Sets as Trees using Rank

Idea:

Union on Disjoint-Sets as Trees using Rank

Idea:

- For every node keep track of its rank which denotes its height in the tree.

Union on Disjoint-Sets as Trees using Rank

Idea:

- For every node keep track of its rank which denotes its height in the tree.
- During **Union**:

Union on Disjoint-Sets as Trees using Rank

Idea:

- For every node keep track of its rank which denotes its height in the tree.
- During **Union**:
  - Root with smaller rank will point to root with larger rank.

Union on Disjoint-Sets as Trees using Rank

Idea:

- For every node keep track of its rank which denotes its height in the tree.
- During **Union**:
  - Root with smaller rank will point to root with larger rank.
  - If roots have the same rank then anyone can point to the other one and rank of the new

Union on Disjoint-Sets as Trees using Rank

Idea:

- For every node keep track of its rank which denotes its height in the tree.
- During **Union**:
  - Root with smaller rank will point to root with larger rank.
  - If roots have the same rank then anyone can point to the other one and rank of the new representative will increase by one.

Union on Disjoint-Sets as Trees using Rank

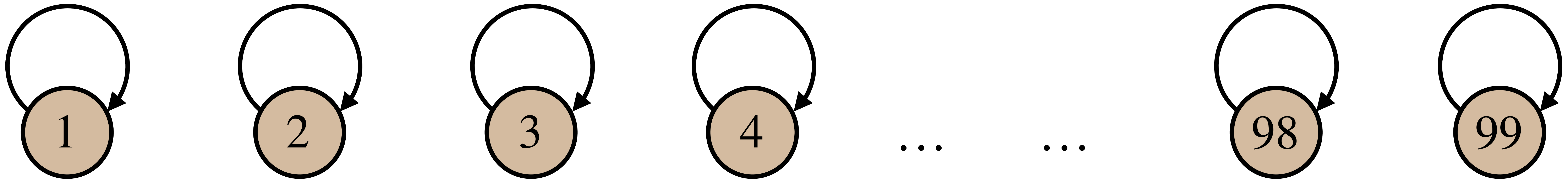
Rank starts with 0

Idea:

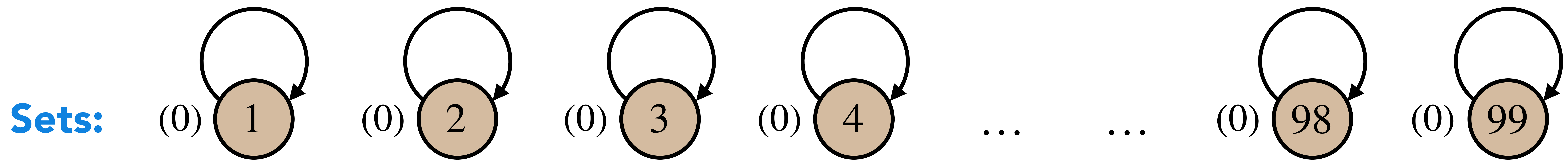
- For every node keep track of its rank which denotes its height in the tree.
- During **Union**:
  - Root with smaller rank will point to root with larger rank.
  - If roots have the same rank then anyone can point to the other one and rank of the new representative will increase by one.

Union on Disjoint-Sets as Trees using Rank

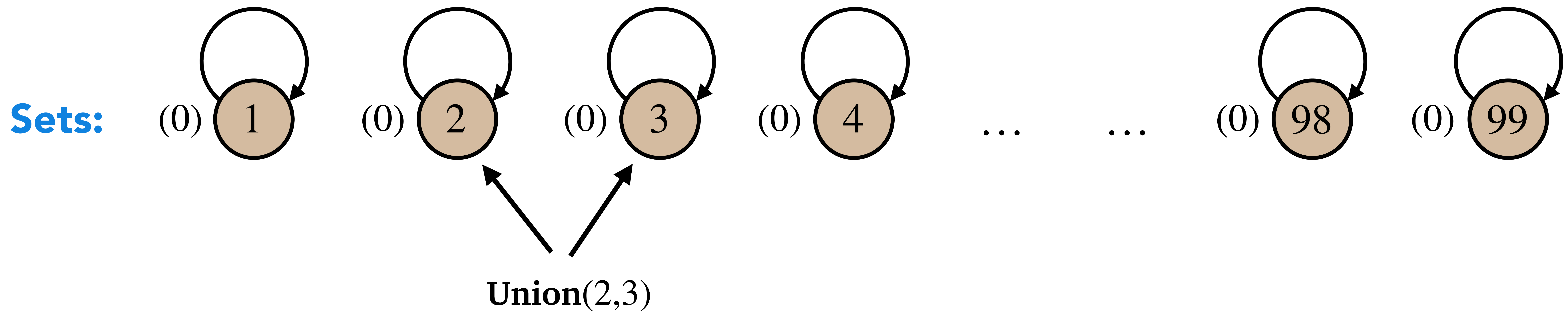
Sets:



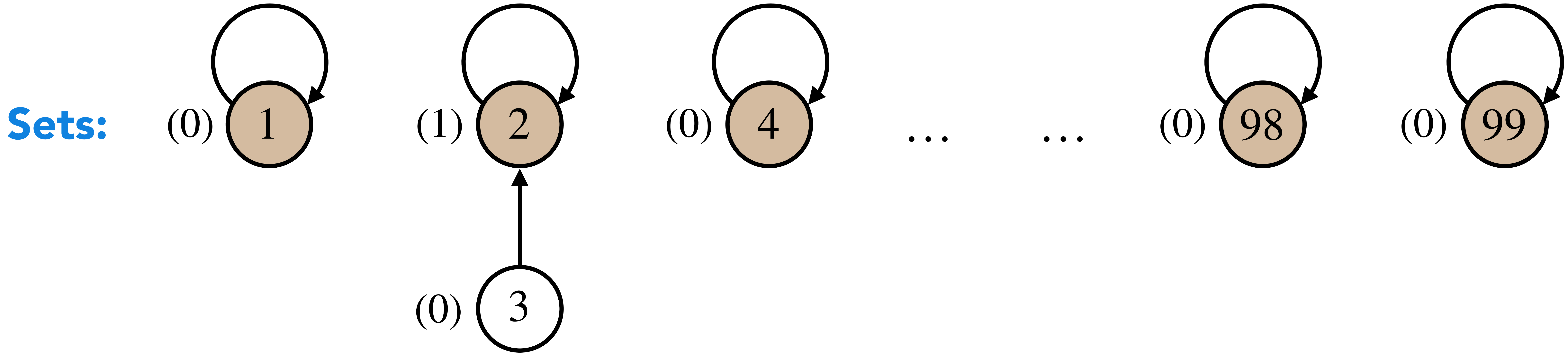
Union on Disjoint-Sets as Trees using Rank



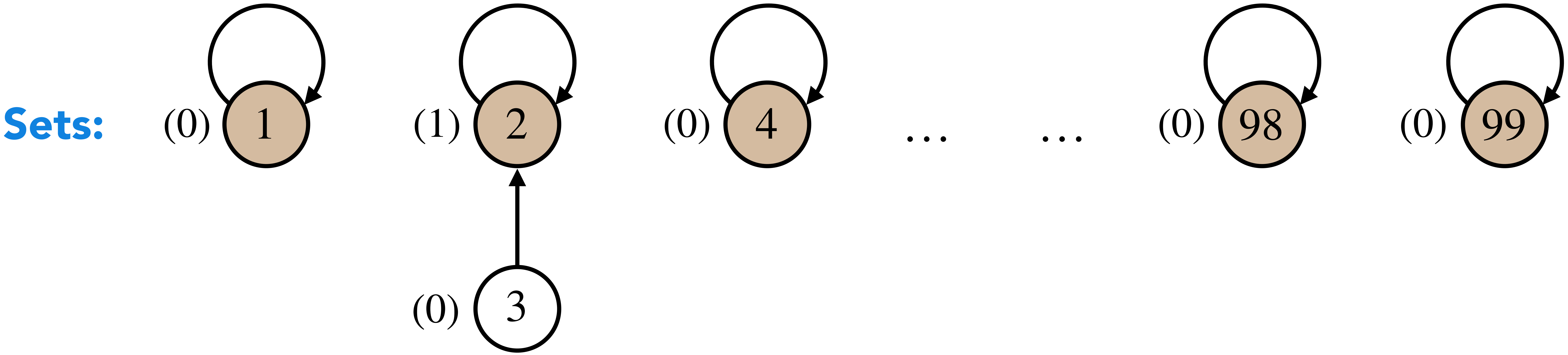
Union on Disjoint-Sets as Trees using Rank



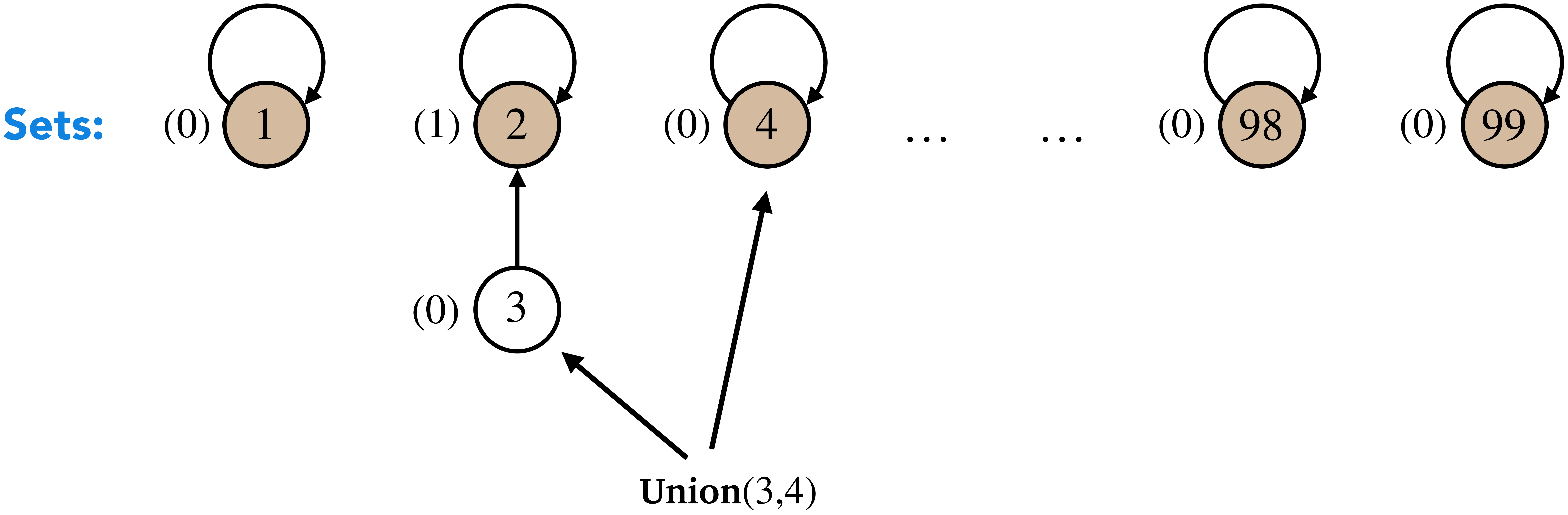
Union on Disjoint-Sets as Trees using Rank



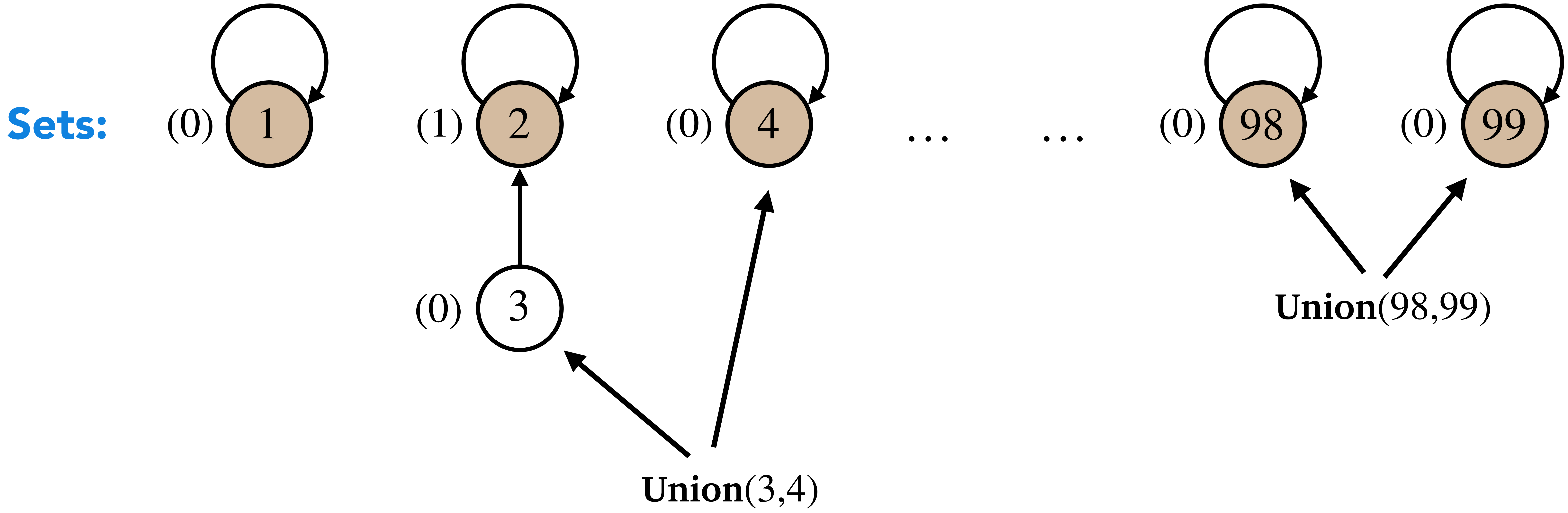
Union on Disjoint-Sets as Trees using Rank



Union on Disjoint-Sets as Trees using Rank

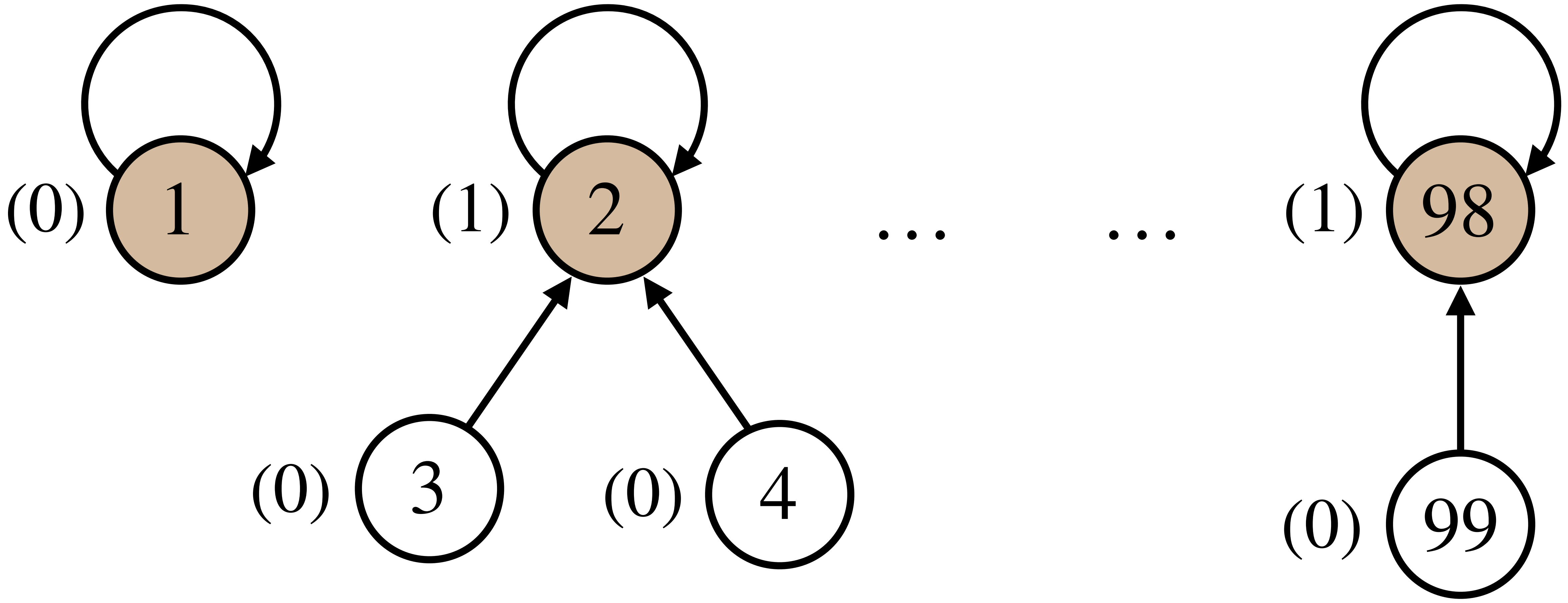


Union on Disjoint-Sets as Trees using Rank

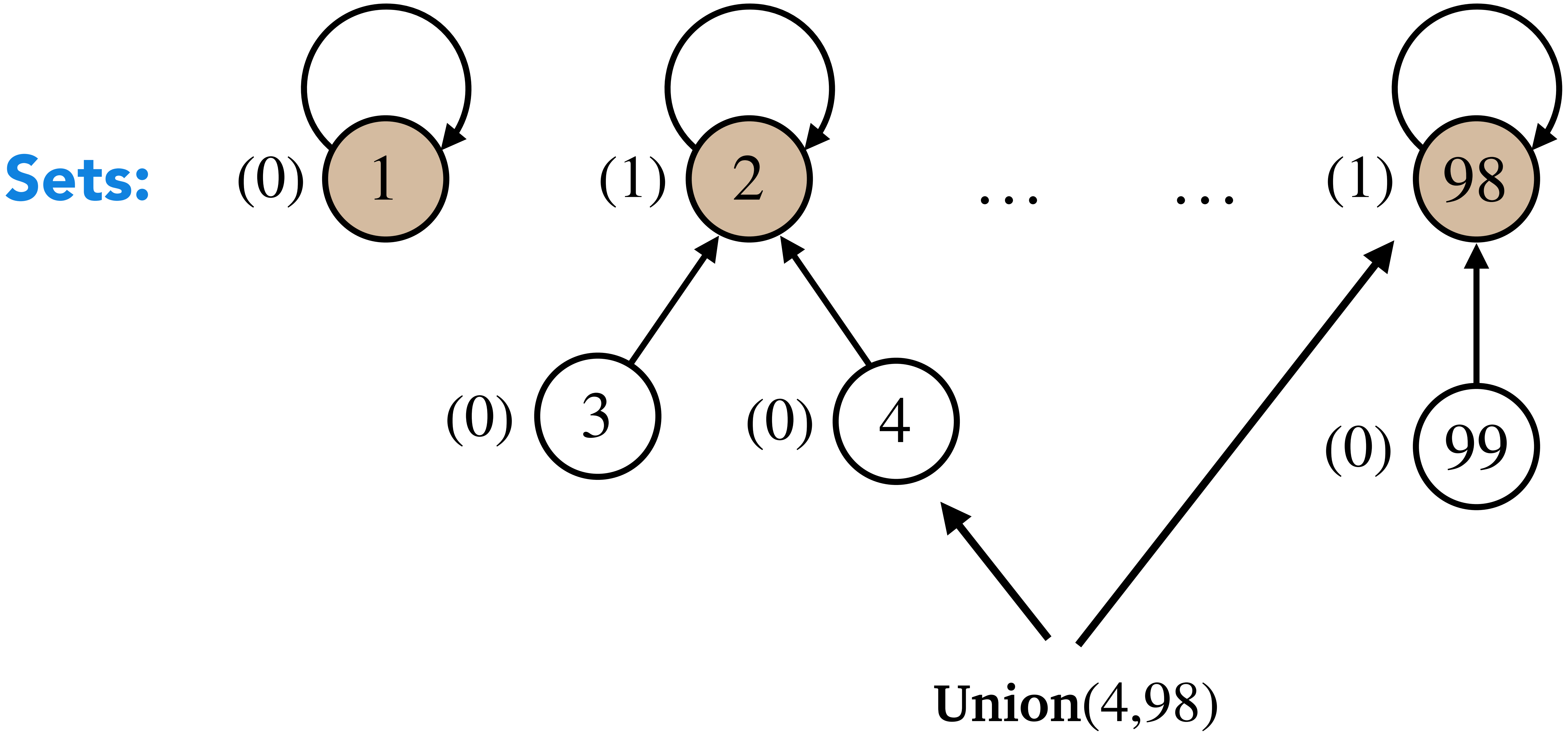


Union on Disjoint-Sets as Trees using Rank

Sets:

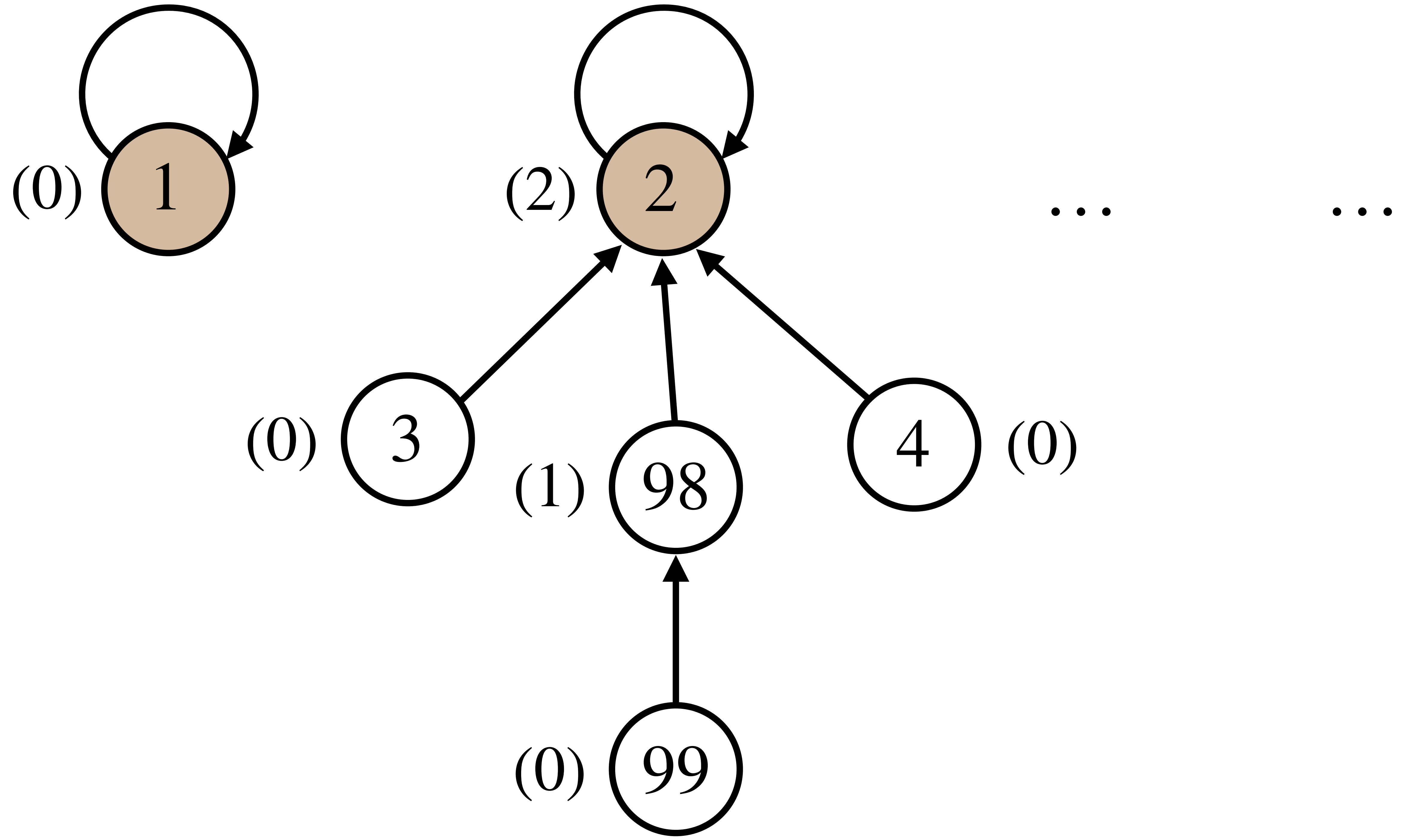


Union on Disjoint-Sets as Trees using Rank



Union on Disjoint-Sets as Trees using Rank

Sets:



Disjoint-Sets as Trees: Operations

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

1. $x.p = x$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

1. $x\,.\,p = x$
2. $x\,.\,rank = 0$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

1. $x.p = x$
2. $x.rank = 0$

Find-Set(x)**:**

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

1. $x.p = x$
2. $x.rank = 0$

Find-Set(x):

1. **if** $x \neq x.p$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

1. $x.p = x$
2. $x.rank = 0$

Find-Set(x)**:**

1. **if** $x \neq x.p$
2. **return Find-Set**$(x.p)$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

1. $x\,.\,p = x$
2. $x\,.\,rank = 0$

Find-Set(x)**:**

1. **if** $x \neq x\,.\,p$
2. **return Find-Set**$(x\,.\,p)$
3. **else**

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

1. $x\,.\,p = x$
2. $x\,.\,rank = 0$

Find-Set(x)**:**

1. **if** $x \neq x\,.\,p$
2.   **return Find-Set**$(x\,.\,p)$
3. **else**
4.   **return** x

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

1. $x.p = x$
2. $x.rank = 0$

Find-Set(x):

1. **if** $x \neq x.p$
2. **return Find-Set**$(x.p)$
3. **else**
4. **return** x

Union(x, y):

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

$x\,.\,p = x$
$x\,.\,rank = 0$

Find-Set(x)**:**

if $x \neq x\,.\,p$
    return Find-Set$(x\,.\,p)$
else
    return x

Union(x, y)**:**

$x =$ **Find-Set**(x), $y =$ **Find-Set**(y)

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

1. $x.p = x$
2. $x.rank = 0$

Find-Set(x):

1. **if** $x \neq x.p$
2. **return Find-Set**$(x.p)$
3. **else**
4. **return** x

Union(x, y):

1. $x =$ **Find-Set**(x), $y =$ **Find-Set**(y)
2. **if** $x.rank > y.rank$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

$x\,.\,p = x$
$x\,.\,rank = 0$

Find-Set(x):

if $x \neq x\,.\,p$
    return Find-Set$(x\,.\,p)$
else
    return x

Union(x, y):

$x =$ **Find-Set**(x), $y =$ **Find-Set**(y)
if $x\,.\,rank > y\,.\,rank$
    $y\,.\,p = x$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

1. $x\,.\,p = x$
2. $x\,.\,rank = 0$

Find-Set(x):

1. **if** $x \neq x\,.\,p$
2.   **return Find-Set**$(x\,.\,p)$
3. **else**
4.   **return** x

Union(x, y):

1. $x =$ **Find-Set**(x), $y =$ **Find-Set**(y)
2. **if** $x\,.\,rank > y\,.\,rank$
3. $y\,.\,p = x$
4. **else if** $x\,.\,rank < y\,.\,rank$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

$x.p = x$

$x.rank = 0$

Find-Set(x):

if $x \neq x.p$

    return Find-Set$(x.p)$

else

    return x

Union(x, y):

$x =$ **Find-Set**(x), $y =$ **Find-Set**(y)

if $x.rank > y.rank$

    $y.p = x$

else if $x.rank < y.rank$

    $x.p = y$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

$x\,.\,p = x$
$x\,.\,rank = 0$

Find-Set(x):

if $x \neq x\,.\,p$
    return Find-Set$(x\,.\,p)$
else
    return x

Union(x, y):

$x =$ **Find-Set**(x), $y =$ **Find-Set**(y)
if $x\,.\,rank > y\,.\,rank$
    $y\,.\,p = x$
else if $x\,.\,rank < y\,.\,rank$
    $x\,.\,p = y$
else

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

$x\,.\,p = x$
$x\,.\,rank = 0$

Find-Set(x)**:**

if $x \neq x\,.\,p$
    return Find-Set$(x\,.\,p)$
else
    return x

Union(x, y)**:**

$x =$ **Find-Set**(x), $y =$ **Find-Set**(y)
if $x\,.\,rank > y\,.\,rank$
    $y\,.\,p = x$
else if $x\,.\,rank < y\,.\,rank$
    $x\,.\,p = y$
else
    $x\,.\,p = y$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x):

$x.p = x$
$x.rank = 0$

Find-Set(x):

if $x \neq x.p$
    return Find-Set$(x.p)$
else
    return x

Union(x, y):

$x =$ **Find-Set**(x), $y =$ **Find-Set**(y)
if $x.rank > y.rank$
    $y.p = x$
else if $x.rank < y.rank$
    $x.p = y$
else
    $x.p = y$
    $y.rank = y.rank + 1$

Disjoint-Sets as Trees: Operations

Recall that we need to perform three operations: **Make-Set**(x), **Union**(x, y), and **Find-Set**(x).

Make-Set(x)**:**

$x.p = x$
$x.rank = 0$

Find-Set(x)**:**

if $x \neq x.p$
    return Find-Set$(x.p)$
else
    return x

Assume x and y are in different sets

Union(x, y)**:**

$x =$ **Find-Set**(x), $y =$ **Find-Set**(y)
if $x.rank > y.rank$
    $y.p = x$
else if $x.rank < y.rank$
    $x.p = y$
else
    $x.p = y$
    $y.rank = y.rank + 1$

Disjoint-Sets as Trees: Analysis

Disjoint-Sets as Trees: Analysis

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations,

Disjoint-Sets as Trees: Analysis

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set**

Disjoint-Sets as Trees: Analysis

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Disjoint-Sets as Trees: Analysis

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Proving rank of any tree (set) can be at most $O(\lg n)$ is sufficient for proving above claim.

Disjoint-Sets as Trees: Analysis

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof:

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step:

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step:

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$. Let x be a node with rank $i + 1$.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$. Let x be a node with rank $i + 1$.

Case 1: The first time when x's rank changed from i to $i + 1$ and it became root of tree, say T,

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$. Let x be a node with rank $i + 1$.

Case 1**:** The first time when x's rank changed from i to $i + 1$ and it became root of tree, say T, it must have been a union of two trees say T_1 and T_2 with their roots' rank i.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$. Let x be a node with rank $i + 1$.

Case 1**:** The first time when x's rank changed from i to $i + 1$ and it became root of tree, say T, it must have been a union of two trees say T_1 and T_2 with their roots' rank i.

From inductive hypothesis each of T_1 and T_2 contain at least 2^i nodes.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$. Let x be a node with rank $i + 1$.

Case 1**:** The first time when x's rank changed from i to $i + 1$ and it became root of tree, say T, it must have been a union of two trees say T_1 and T_2 with their roots' rank i.

From inductive hypothesis each of T_1 and T_2 contain at least 2^i nodes.
Hence, $T = T_1 \cup T_2$ will contain at least $2^i + 2^i = 2^{i+1}$ nodes.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$. Let x be a node with rank $i + 1$.

Case 1**:** The first time when x's rank changed from i to $i + 1$ and it became root of tree, say T, it must have been a union of two trees say T_1 and T_2 with their roots' rank i.

From inductive hypothesis each of T_1 and T_2 contain at least 2^i nodes.
Hence, $T = T_1 \cup T_2$ will contain at least $2^i + 2^i = 2^{i+1}$ nodes.

Case 2**:** At rank $i + 1$ of x, every **union** operation can add $k \geq 0$ nodes in $tree(x)$.

Disjoint-Sets as Trees: Analysis

Claim: A node with rank (or height) h has at least 2^h nodes in the subtree rooted at that node.

Proof: We will prove it using induction on h.

Basis Step: Nodes in subtree of a node with rank 0 contains 1 node. Trivially true.

Inductive Step: Assuming the claim is true for nodes with rank $\leq i$, we will prove it for nodes with rank $i + 1$. Let x be a node with rank $i + 1$.

Case 1**:** The first time when x's rank changed from i to $i + 1$ and it became root of tree, say T, it must have been a union of two trees say T_1 and T_2 with their roots' rank i.

From inductive hypothesis each of T_1 and T_2 contain at least 2^i nodes.
Hence, $T = T_1 \cup T_2$ will contain at least $2^i + 2^i = 2^{i+1}$ nodes.

Case 2**:** At rank $i + 1$ of x, every **union** operation can add $k \geq 0$ nodes in $tree(x)$.

■

Disjoint-Sets as Trees: Analysis

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof:

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ∎

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ∎

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations,

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ■

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set**

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ■

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ■

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Proof:

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ■

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Proof: **Make-Set** operations take constant time.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ■

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Proof: **Make-Set** operations take constant time.

Union operations take the same time as **Find-Set**.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ■

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Proof: **Make-Set** operations take constant time.

Union operations take the same time as **Find-Set**.

Find-Set operations take $O(h)$ time, where $h \leq \lfloor \lg n \rfloor$ is the rank of the root of the tree.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ∎

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Proof: **Make-Set** operations take constant time.

Union operations take the same time as **Find-Set**.

Find-Set operations take $O(h)$ time, where $h \leq \lfloor \lg n \rfloor$ is the rank of the root of the tree.

Hence, m operations take $O(m \lg n)$ time.

Disjoint-Sets as Trees: Analysis

Claim: Every node has rank at most $\lfloor \lg n \rfloor$ in the disjoint-set via trees using rank heuristic.

Proof: Suppose a node has rank $\lfloor \lg n \rfloor + k$, where $k > 0$.

Then, from the previous claim its subtree should contain at least $2^{\lfloor \lg n \rfloor + k}$ nodes.

But, $2^{\lfloor \lg n \rfloor + k} > n$, which is not possible. ■

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m \lg n)$ time in the tree using rank implementation.

Proof: **Make-Set** operations take constant time.

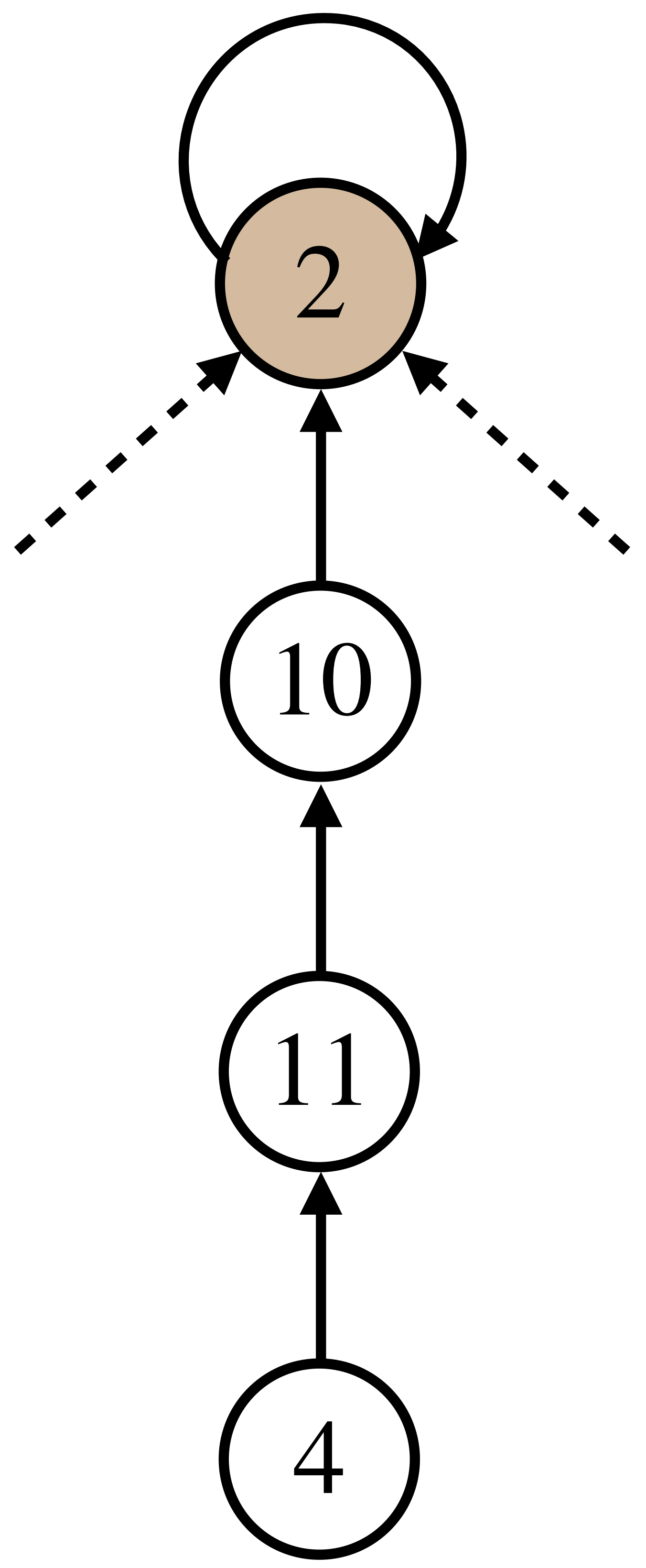
Union operations take the same time as **Find-Set**.

Find-Set operations take $O(h)$ time, where $h \leq \lfloor \lg n \rfloor$ is the rank of the root of the tree.

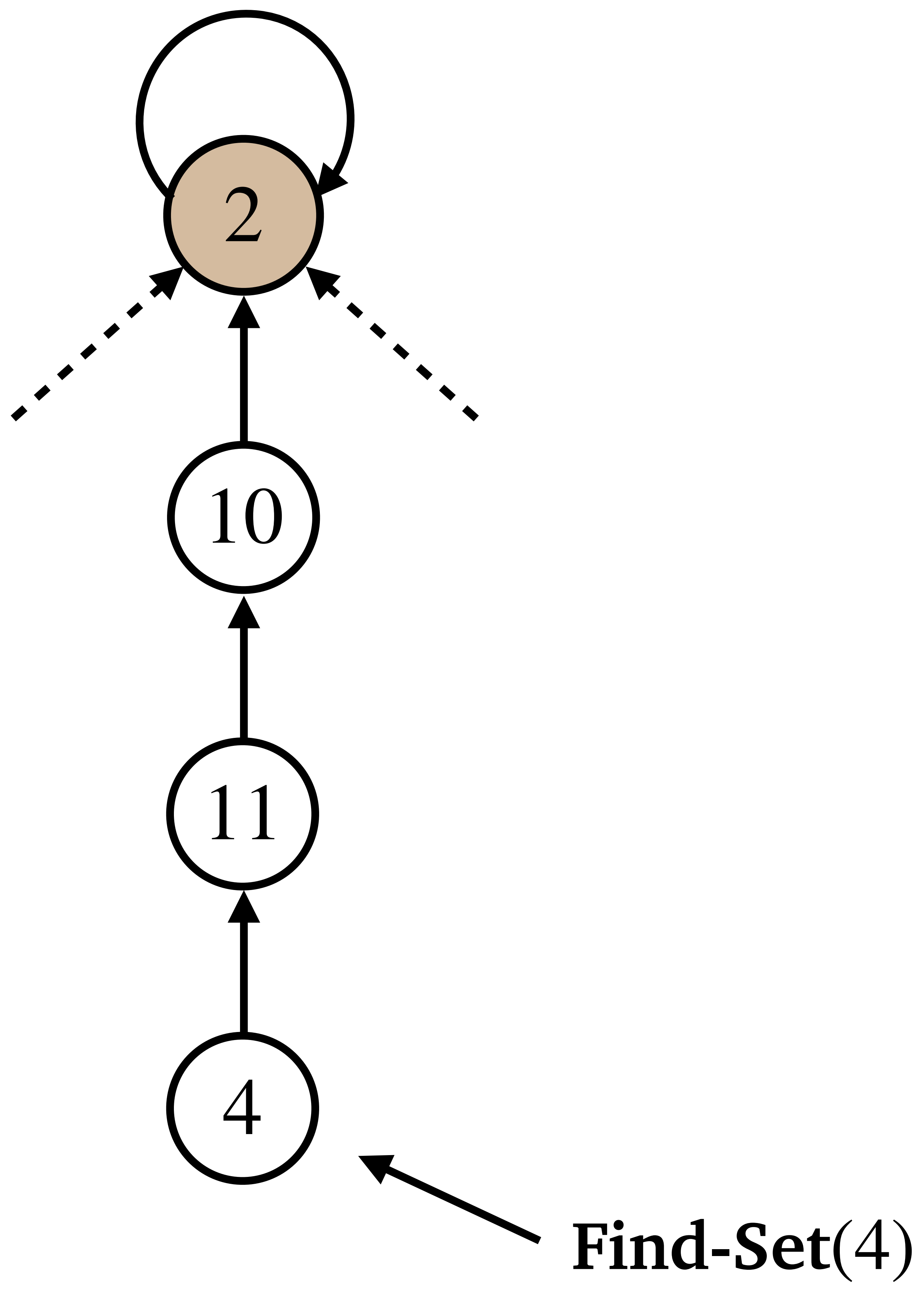
Hence, m operations take $O(m \lg n)$ time. ■

Path-Compression Heuristic

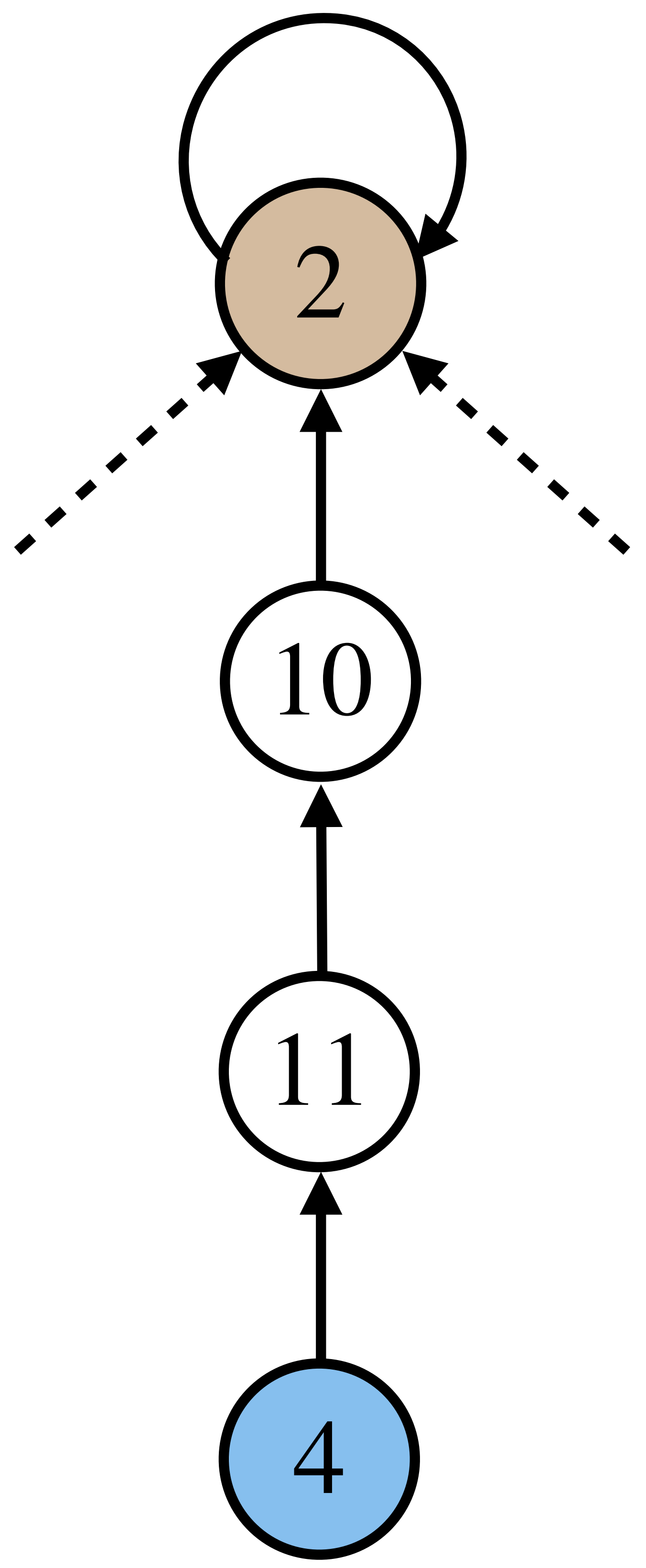
Path-Compression Heuristic



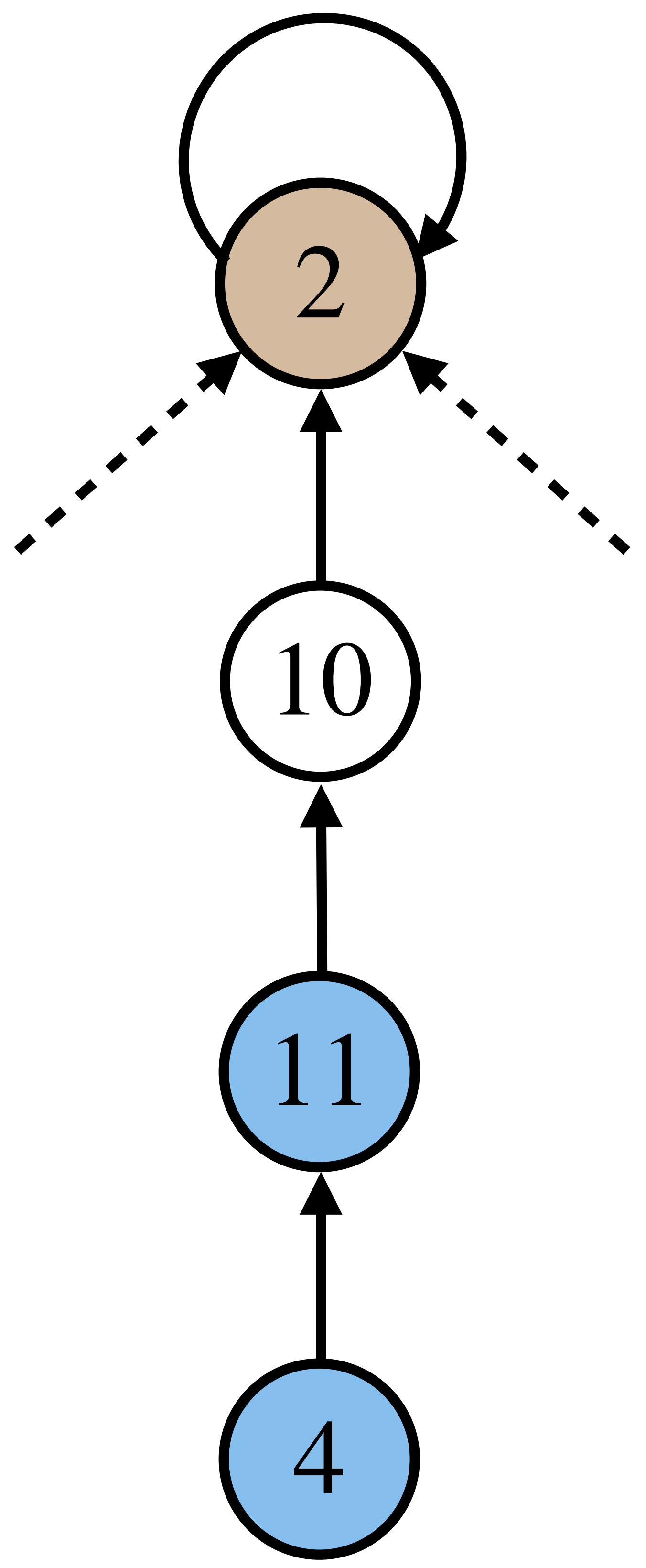
Path-Compression Heuristic



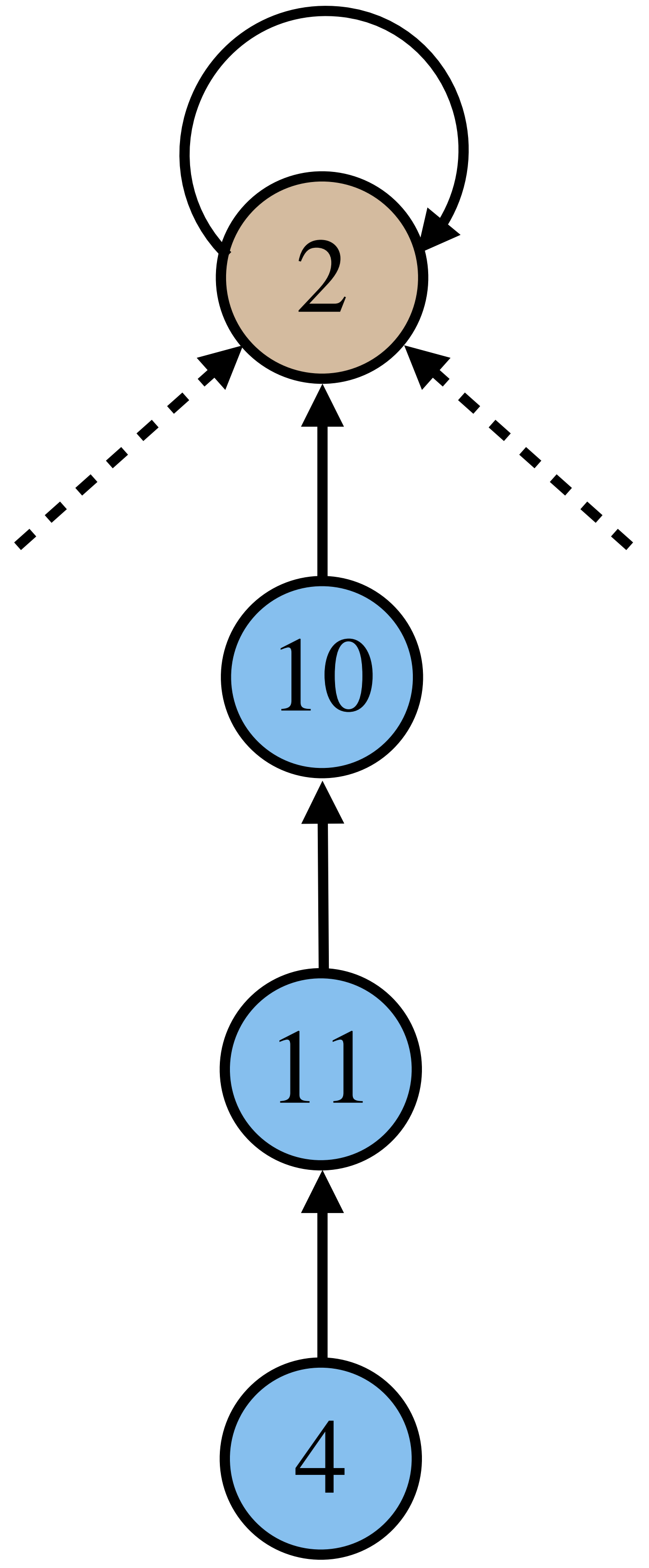
Path-Compression Heuristic



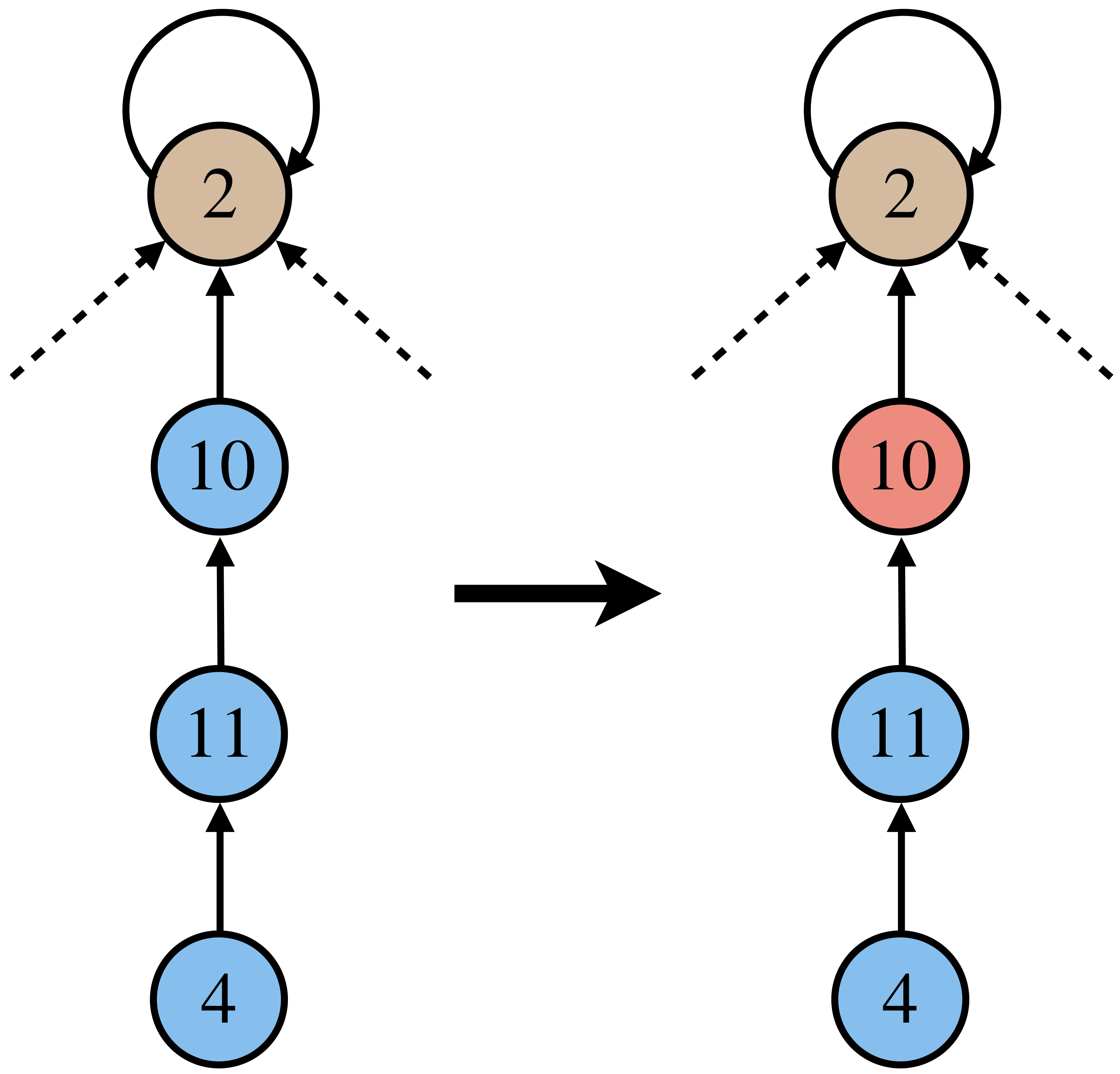
Path-Compression Heuristic



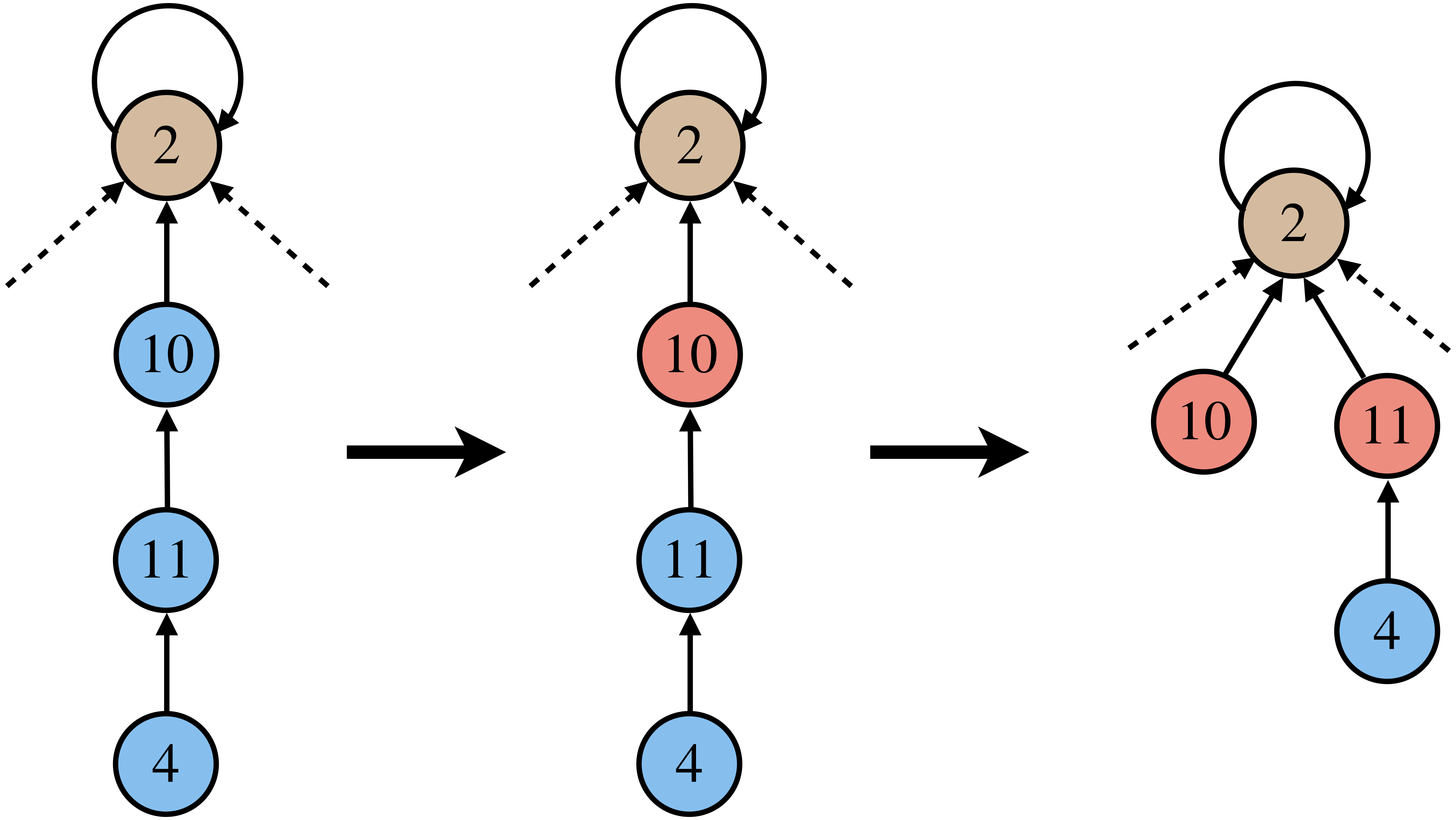
Path-Compression Heuristic



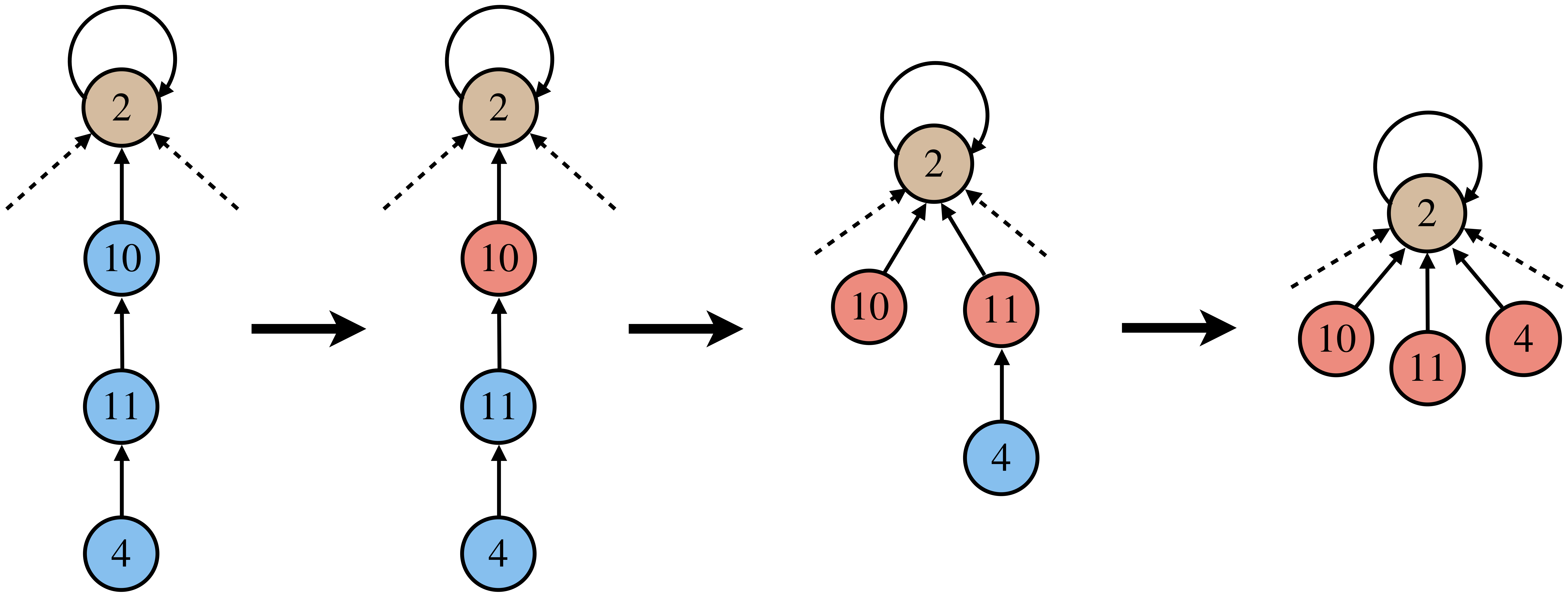
Path-Compression Heuristic



Path-Compression Heuristic

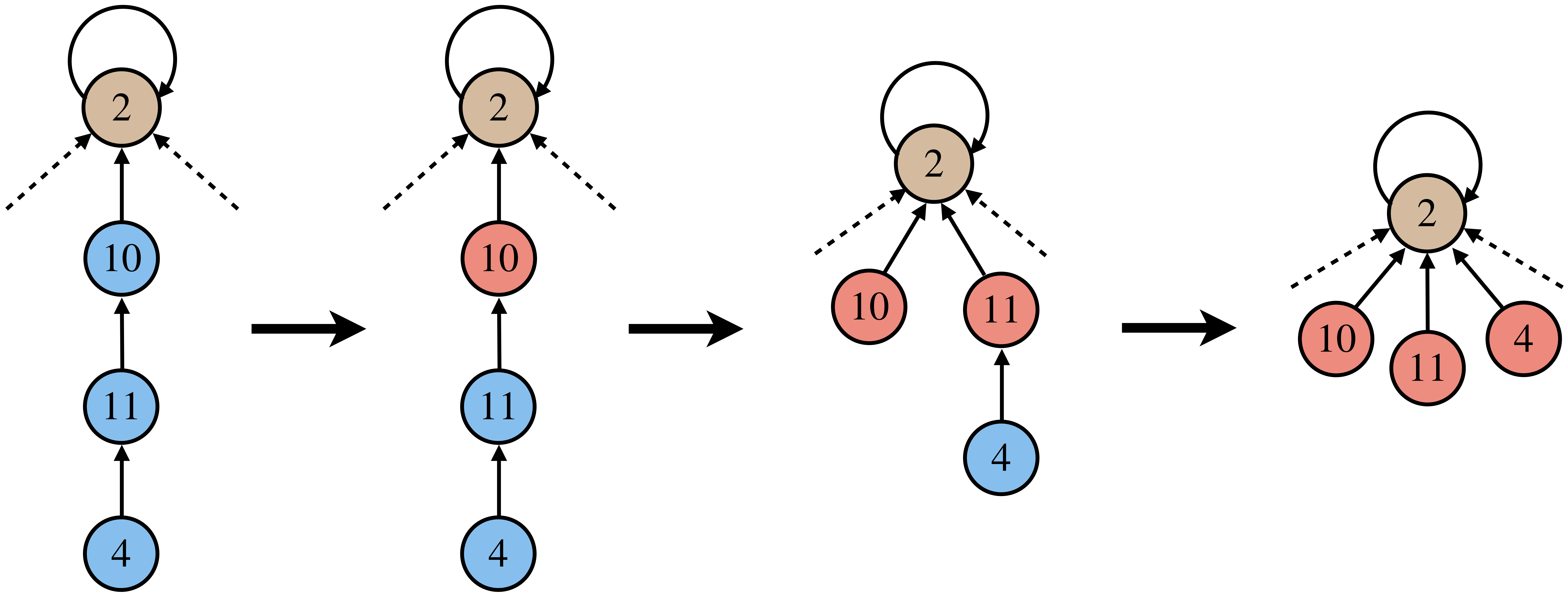


Path-Compression Heuristic



Path-Compression Heuristic

In Path-Compression, while performing **Find-Set**(x) we make root the parent of every node on path from x to root.



Disjoint-Sets as Trees: Operations

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

1. **if** $x \neq x.p$

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

1. **if** $x \neq x\,.\,p$
2. **return Find-Set**($x\,.\,p$)

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

1. **if** $x \neq x.p$
2. **return Find-Set**($x.p$)
3. **else**

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

if $x \neq x.p$

  return Find-Set($x.p$)

else

  return x

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

```
Find-Set(x):
    if x ≠ x.p
        return Find-Set(x.p)
    else
        return x
```

Old **Find-Set**(x)

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

1. **if** $x \neq x\,.\,p$
2.   **return Find-Set**($x\,.\,p$)
3. **else**
4.   **return** x

Old **Find-Set**(x)

→

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

```
if x ≠ x.p
    return Find-Set(x.p)
else
    return x
```

Old **Find-Set**(x)

→

Find-Set(x):

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

1. **if** $x \neq x.p$
2. **return Find-Set**($x.p$)
3. **else**
4. **return** x

Old **Find-Set**(x)

→

Find-Set(x):

1. **if** $x \neq x.p$

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

1. **if** $x \neq x\,.\,p$
2. **return Find-Set**($x\,.\,p$)
3. **else**
4. **return** x

Old **Find-Set**(x)

→

Find-Set(x):

1. **if** $x \neq x\,.\,p$
2. $x\,.\,p =$ **Find-Set**($x\,.\,p$)

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

1. **if** $x \neq x.p$
2.   **return Find-Set**($x.p$)
3. **else**
4.   **return** x

Old **Find-Set**(x)

→

Find-Set(x):

1. **if** $x \neq x.p$
2. $x.p =$ **Find-Set**($x.p$)
3. **return** $x.p$

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

    if $x \neq x\,.\,p$
        return Find-Set($x\,.\,p$)
    else
        return x

Old **Find-Set**(x)

Find-Set(x):

    if $x \neq x\,.\,p$
        $x\,.\,p =$ **Find-Set**($x\,.\,p$)
    return $x\,.\,p$

Find-Set(x) with path-compression

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

if $x \neq x\,.\,p$
  return Find-Set$(x\,.\,p)$
else
  return x

Old **Find-Set**(x)

Find-Set(x):

if $x \neq x\,.\,p$
  $x\,.\,p =$ **Find-Set**$(x\,.\,p)$
return $x\,.\,p$

Find-Set(x) with path-compression

Note: When using path-compression heuristic, rank gives an upper bound on the height of a

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

```
Find-Set(x):
    if x ≠ x.p
        return Find-Set(x.p)
    else
        return x
```

Old **Find-Set**(x)

```
Find-Set(x):
    if x ≠ x.p
        x.p = Find-Set(x.p)
    return x.p
```

Find-Set(x) with path-compression

Note: When using path-compression heuristic, rank gives an upper bound on the height of a node.

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

if $x \neq x\,.\,p$
    return Find-Set$(x\,.\,p)$
else
    return x

Old **Find-Set**(x)

Find-Set(x):

if $x \neq x\,.\,p$
    $x\,.\,p =$ **Find-Set**$(x\,.\,p)$
return $x\,.\,p$

Find-Set(x) with path-compression

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

if $x \neq x\,.\,p$
    return Find-Set$(x\,.\,p)$
else
    return x

Old **Find-Set**(x)

Find-Set(x):

if $x \neq x\,.\,p$
    $x\,.\,p =$ **Find-Set**$(x\,.\,p)$
return $x\,.\,p$

Find-Set(x) with path-compression

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations,

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

Find-Set(x):

if $x \neq x\,.\,p$
    return Find-Set$(x\,.\,p)$
else
    return x

Old **Find-Set**(x)

Find-Set(x):

if $x \neq x\,.\,p$
    $x\,.\,p =$ **Find-Set**$(x\,.\,p)$
return $x\,.\,p$

Find-Set(x) with path-compression

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set**

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

```
Find-Set(x):
    if x ≠ x.p
        return Find-Set(x.p)
    else
        return x
```

Old **Find-Set**(x)

```
Find-Set(x):
    if x ≠ x.p
        x.p = Find-Set(x.p)
    return x.p
```

Find-Set(x) with path-compression

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m\alpha(n))$ time using **rank** and **path-compression** heuristic.

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

```
Find-Set(x):
    if x ≠ x.p
        return Find-Set(x.p)
    else
        return x
```

Old **Find-Set**(x)

```
Find-Set(x):
    if x ≠ x.p
        x.p = Find-Set(x.p)
    return x.p
```

Find-Set(x) with path-compression

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m\alpha(n))$ time using **rank** and **path-compression** heuristic.

Inverse of Ackerman function, $\alpha(n)$, is a very very slowly growing function.

Disjoint-Sets as Trees: Operations

Change **Find-Set**(x) to implement path-compression.

```
Find-Set(x):
    if x ≠ x.p
        return Find-Set(x.p)
    else
        return x
```

Old **Find-Set**(x)

```
Find-Set(x):
    if x ≠ x.p
        x.p = Find-Set(x.p)
    return x.p
```

Find-Set(x) with path-compression

Claim: A sequence of m **Make-Set**, **Union**, & **Find-Set** operations, first n of which are **Make-Set** operations, takes $O(m\alpha(n))$ time using rank and path-compression heuristic.

$\alpha(n) \leq 4$ for $n \leq 10^{80}$.